

LORENTZ-COVARIANT ANALYSIS OF A QUANTUM SOLITON

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Abstract

A set of integral relations for rotational and translational zero modes in the vicinity of the classical soliton solution are derived from the particle-like properties of the latter. The validity of these all relations is considered for a number of soliton models in 2+1- and 3+1-dimensions.

Theories with non-trivial classical solutions, such as the skyrmion models of baryons [1], are object of intensive investigations. Quantization of hedgehog-type configurations including translational and rotational degrees of freedom [2] leads to the quantum Hamiltonian which contains a full bilinear form in conjugated momenta with nontrivial couplings between different collective variables [3–5]. However, for a particle-like classical solution one should expect additive diagonal contributions of kinetic and centrifugal terms to the Hamiltonian, at least to the lowest orders in the appropriate weak coupling expansion. Actually, it is a part of the general problem of decoupling upon quantization of various soliton degrees of freedom, which takes place for any type of field models with classical solutions. In this report we'll present a consistent general analysis of this problem, based on the particle-like properties of the classical solution combined with Lorentz covariance and virial relations. In the present report, we also give an analysis for the soliton with spin, quantized by means of translational and rotational collective coordinates, into corresponding representation of the Poincaré group.

Let us consider a field theory in $d + 1$ space-time dimensions described by the Lagrangian density $\mathcal{L}(\varphi)$, which possesses a classical particle-like solution $\varphi_c(x)$. It is generally accepted, that if in the rest frame φ_c is static with finite and localized

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energy density, then in quantum version of the theory such configuration describes an extended particle. Now we'll show, that there exists a set of nontrivial integral relations, fulfilled by $\varphi_c(x)$, which provide the validity of these assumptions.

For a given static solution $\varphi_c(\vec{x})$ the moving one is constructed via Lorentz boost, what results in the replacement

$$x^i \rightarrow \Lambda^{-1i}_{\nu} x^{\nu} \quad (1)$$

in the arguments of φ_c , where Λ_{ν}^{μ} is the corresponding Lorentz matrix. The momentum of the moving solution is

$$P^{\mu} = \int T^{\mu 0}(\varphi_c(\vec{x}, x^0)) d\vec{x}, \quad (2)$$

where $T^{\mu\nu}(\varphi_c)$ is the energy-momentum tensor. Transforming the r.h.s. in (2) to the rest frame, one gets

$$P^{\mu} = \Lambda_{\mu'}^{\mu} \Lambda_{\nu'}^0 \int T^{\mu'\nu'}(\varphi_c(\vec{\xi})) J d\vec{\xi}, \quad (3)$$

where $J = \Lambda_0^{0-1}$ is the Jacobian of transition from $d\vec{x}$ to the rest frame spatial variable. On the other hand, the l.h.s. of (2) should be the momentum of a particle with the mass M , that is

$$P^{\mu} = \Lambda_0^{\mu} M. \quad (4)$$

From eqs. (3) and (4) for $\nu = 0$ we get $P^0 = M$, $P^i = 0$, just that we should expect for a static solution. However for $\mu = i$, $\nu = j$ we obtain

$$\int \frac{\partial \mathcal{L}(\varphi_c)}{\partial \partial^j \varphi_c(\vec{\xi})} \partial_i \varphi_c(\vec{\xi}) d\vec{\xi} = M \delta_{ij}. \quad (5)$$

So we get the first set of conditions (5), which holds for a particle-like classical configuration $\varphi_c(\vec{\xi})$ in the rest frame.

Now let us consider the orbital part of the 4-rotation tensor (without the spin term)

$$L^{\mu\nu} = \int (x^{\nu} T^{\mu 0}(\varphi_c(\vec{x}, x^0)) - x^{\mu} T^{\nu 0}(\varphi_c(\vec{x}, x^0))) d\vec{x}. \quad (6)$$

Analogous calculations leads to the following relations (for definiteness, we take $d = 3$)

$$\int \varepsilon_{lij} \xi_i \partial_j \varphi_c(\vec{\xi}) \frac{\partial \mathcal{L}(\varphi_c)}{\partial \partial^k \varphi_c(\vec{\xi})} d\vec{\xi} = 0, \quad (7)$$

since L^{0i} vanish in the rest frame by assumption. This is the second set of relations on $\varphi_c(\vec{\xi})$, following from the Lorentz covariance and particle-likeness of the classical solution.

So each particle-like solution should be subject of conditions (5) and (7). It should be noted, that the relation (4) for $\mu = 0$ reproduces nothing else but the

relativistic mass-energy relation. For the moving φ^4 -kink solution this relation has been explicitly verified in [5], and for the moving skyrmion — in [6]) by direct calculations. However, the eqs. (5) are more general and, moreover, the eqs. (7) also take place. Note also, that these relations, being consistent with the field equations and conservation laws, do not be the direct consequences of the latters, and should be considered separately.

As a direct result of these relations we get the orthogonality of the zero-frequency eigenfunctions in the neighborhood of the classical particle-like solution [7]. Let us discuss the theory of a nonlinear scalar field in 3 spatial dimensions, described by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - U(\varphi), \quad (8)$$

which possesses a static soliton solution

$$\varphi_c(x) = u(\vec{x}). \quad (9)$$

In the general case the non-spherical configuration $u(\vec{x})$ yields 6 zero-frequency modes — three translational ones $\psi_i(\vec{x}) = \partial_i u(\vec{x})$ and three rotational $f_i(\vec{x}) = \varepsilon_{ijk} x_j \partial_k u(\vec{x})$. Then from eqs. (5) and (7) one immediately obtains

$$\int d\vec{\xi} \psi_i(\vec{\xi}) \psi_j(\vec{\xi}) = M \delta_{ij}, \quad (10)$$

$$\int d\vec{\xi} f_i(\vec{\xi}) \psi_j(\vec{\xi}) = 0. \quad (11)$$

Further, by spatial rotations one can always achieve that

$$\int d\vec{\xi} f_i(\vec{\xi}) f_j(\vec{\xi}) = \Omega_{ij} = \Omega_i \delta_{ij}, \quad (12)$$

where Ω_i are the moments of inertia of the classical configuration. Obviously, the relations (10) and (11) remain unchanged.

So the particle-likeness of the classical solution results in the diagonality of the zero-frequency scalar product matrix. This diagonality plays an essential role in the procedure of quantization in the vicinity of a classical soliton solution by means of collective coordinates [3, 5]. Following the conventional procedure [4], let us consider the field $\varphi(\vec{x})$ in the Schrödinger picture in the vicinity of the solution $u(\vec{x})$. The substitution, introducing translational and rotational collective coordinates, reads [7]

$$\varphi(\vec{x}) = u\left(R^{-1}(\vec{c})(\vec{x} - \vec{q})\right) + \Phi\left(R^{-1}(\vec{c})(\vec{x} - \vec{q})\right), \quad (13)$$

where Φ is the meson field, $R(\vec{c})$ is the rotation matrix, \vec{q} and \vec{c} are the translational and rotational collective coordinates correspondingly.

In order to keep the number of degrees of freedom we impose on the field $\Phi(\vec{y})$ 6 subsidiary conditions, which in the theory of a weak coupling are usually taken as linear combinations

$$\int d\vec{y} N^{(\alpha)}(\vec{y}) \Phi(\vec{y}) = 0, \quad \alpha = 1, \dots, 6. \quad (14)$$

The set $\{N^{(\alpha)}(\vec{y})\}$ should ensure the condition of orthogonality of the meson field $\Phi(\vec{y})$ to zero-frequency modes and is chosen as [8] $N^{(\alpha)}(\vec{y}) = \{\psi_i(\vec{y})/M, f_i(\vec{y})/\Omega_i\}$. It is this relation, that ensures the additive form of the collective coordinate part of the Hamiltonian within the weak coupling expansion in powers of the meson fields. Considering the condition (14) as relation, defining \vec{q} and \vec{c} as functionals of $\varphi(\vec{x})$ and calculating the conjugate momentum $\pi(\vec{x}) = -i\frac{\delta}{\delta\varphi(\vec{x})}$ as a composite derivative, we can obtain finally for the Hamiltonian the following lowest-order expression

$$H = M + \frac{\vec{K}^2}{2M} + \frac{1}{2} \sum_i \frac{I_i^2}{\Omega_i}. \quad (15)$$

In eq. (15) \vec{K} and \vec{I} are the momentum and the spin of the field, corresponding to the rotating frame (for details see ref. [8]).

It is indeed such form of the Hamiltonian, that provides to interpret the resulting ground state as non-relativistic particle with the mass M and moments of inertia Ω_i . So the correct form of the Hamiltonian with additive kinetic and centrifugal terms, that means the absence of correlations between translational and rotational degrees of freedom, is ensured by the diagonality of zero-frequency scalar product matrix (10)–(12). In turn, this is a direct consequence of relations (5) and (7). Note also, that this result will be actually valid for any field model in the neighborhood of the suitable soliton solution.

These general considerations can be easily illustrated by concrete models. Firstly, we consider the theory of a scalar field in 1+1-dimensions, described by the Lagrangian density (8). In this case we have only one relation (5)

$$\int dx (\varphi'(x))^2 = M, \quad (16)$$

where the mass M is given by

$$M = \int dx \frac{1}{2}(\varphi'(x))^2 + \int dx U(\varphi(x)). \quad (17)$$

Performing the dilatation $\varphi(x) \rightarrow \varphi(\lambda x)$ and demanding for the solution at $\lambda = 1$, i. e. $\left(\frac{dM(\lambda)}{d\lambda}\right)_{\lambda=1} = 0$, we find the well-known Hobart–Derrick virial relation

$$\frac{1}{2} \int dx (\varphi'(x))^2 = \int dx U(\varphi(x)), \quad (18)$$

owing to which the “particle-likeness condition” (16) is fulfilled automatically.

In 2+1-dimensions, the solitons in CP_N -models are interesting examples with such particle-like properties. As it is well-known, for $N = 1$ the CP_N -model is reduced to $O(3)$ -model, described by

$$\mathcal{L} = \frac{1}{2} \partial_\mu \varphi^a \partial^\mu \varphi^a, \quad \varphi^a \varphi^a = 1. \quad (19)$$

The standard one-particle solution of the model is given by

$$\varphi^1 = \phi(r) \cos n\vartheta, \quad \varphi^2 = \phi(r) \sin n\vartheta, \quad \varphi^3 = (1 - \phi^2)^{1/2}, \quad (20)$$

where r, ϑ are polar coordinates and $\phi(r) = \frac{4r^n}{r^{2n+4}}$, and describes the “baby-skyrmion” configuration with the topological charge $Q = n$ and the mass $M = 4\pi Q$. Inserting the expression (20) into conditions (5) and (7) we obtain, that the conditions of particle-likeness for the solution (20) are satisfied.

As a more nontrivial example, we consider the $SU(2)$ -Skyrme model in 3+1-dimensions [1], including the break-symmetry pion mass term

$$\mathcal{L} = -\frac{f_\pi^2}{4} \text{tr} L_\mu^2 + \frac{1}{32g^2} \text{tr} [L_\mu L_\nu]^2 + \frac{m_\pi^2}{4} \text{tr} (U + U^\dagger - 2), \quad (21)$$

where, as usually, $L_\mu = U^{-1} \partial_\mu U$ is the left chiral current and $U = \sigma + i\tau^a \pi^a$ is the quaternion field. Supposing the conventional “hedgehog” *Ansatz*

$$\sigma = \cos \phi(r), \quad \pi^a = \frac{r^a}{r} \sin \phi(r) \quad (22)$$

we find that the first particle-likeness condition for the skyrmion is fulfilled due to virial relation. Further, inserting the substitution (22) into eqs. (7) we find in the same way, that the second set of relations for the skyrmion is provided by the symmetry properties. So we’ll get upon quantization, that the full bilinear form considered in [2], automatically simplifies up to a diagonal construction similar to eq. (15), and therefore the quantized skyrmion describes an extended non-relativistic particle.

Finally, we consider the ’t Hooft–Polyakov monopole for the $SU(2)$ -Yang–Mills–Higgs theory, described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} (F_{\mu\nu}^a)^2 + \frac{1}{2} (D_\mu \phi^a)^2 - V(\phi) \quad (23)$$

with the monopole solution

$$\phi^a = \frac{1}{g} \frac{r^a}{r^2} H(r), \quad A_i^a = \frac{1}{g} \varepsilon_{aij} \frac{r_j}{r^2} (1 - K(r)), \quad A_0^a = 0. \quad (24)$$

We can find that the particle-likeness condition (5) is fulfilled due to virial relation, and the condition (7) is fulfilled due to symmetry properties of the expression (24), just as in the case of skyrmion.

So we have proved the validity of the “particle-likeness” conditions (5) and (7) for the most important soliton solutions. Note, that there is a close connection between the condition (5) and virial relations for the static configuration by homogeneous dilatations. The relations (7) are usually fulfilled on account of symmetry properties of classical solutions.

Now we give an analysis for the soliton with spin, into corresponding representation of the Poincaré group.

As a first step, we consider a nonlinear scalar field in 3 spatial dimensions, described by the Lagrangian density (8) which possesses a static soliton solution (9). According to the virial theorem, such solutions are unstable in more than one spatial dimension, but for our purposes it is not so important compared to simplicity of presentation. The angular momenta J^i and Lorentz boosts K^i are given by

$$J^i = \int d\vec{x} \varepsilon_{ijk} x^j T^{k0} = \int d\vec{x} \varepsilon_{ijk} x_j \partial_k \varphi \pi. \quad (25)$$

$$K^i = \int d\vec{x} (x^0 T^{i0} - x^i T^{00}) = x^0 P^i - \int d\vec{x} x^i \mathcal{H}, \quad (26)$$

In these expressions $P^i = \int d\vec{x} T^{i0}$ are the spatial momenta, $\mathcal{H} = T^{00} = \pi\dot{\varphi} - \mathcal{L}(\varphi)$ is the Hamiltonian density and $\pi = \partial\mathcal{L}/\partial\dot{\varphi}$ is the canonical field momentum.

Now we put the quasiclassical soliton field $u(R^{-1}(\vec{x} - \vec{q}))$ and its canonical momentum in the leading quasiclassical approximation (for details see ref. [11]) into corresponding Noether expressions for Lorentz generators (25), (26) and demand for their coincidence with the corresponding one-particle representation of the Poincaré group with the same mass M and spin S . It means, that the Lorentz generators $J^{\mu\nu}$ should take the form [9]

$$J^i = \varepsilon_{ijk} q^j P^k + S^i, \quad (27)$$

$$K^i = q^0 P^i - q^i P^0 - \frac{\varepsilon_{ijk} P^j S^k}{P^0 + M}. \quad (28)$$

Firstly, it is a trivial task to verify, that inserting into eq. (25) the soliton operators, one gets identically the eq. (27), provided by the orthogonality conditions (10)–(12).

Applying the same procedure to the Lorentz boost operators (26), we find that the final result is the following set of subsidiary conditions imposed on $u(\vec{x})$

$$\int d\vec{\xi} \xi_i \psi_j(\vec{\xi}) \psi_k(\vec{\xi}) = 0, \quad (29)$$

$$\int d\vec{\xi} \xi_i f_j(\vec{\xi}) f_k(\vec{\xi}) = 0, \quad (30)$$

$$\int d\vec{\xi} \xi_i \psi_j(\vec{\xi}) f_k(\vec{\xi}) = \frac{1}{2} \varepsilon_{ijl} \Omega_{lk}. \quad (31)$$

These relations can be understood as a criterion of “particle-likeness” for the classical soliton field, describing a spinning particle. It should be noted, that whereas the orthogonality conditions (10)–(12) are valid for any static classical solution due to the general properties of lorentz-covariance and so are automatically consistent with equations of motion, the relations (29)–(31) are more strong and restrictive. Namely, the eqs. (11) and (12) are the direct consequences from eqs. (29) and (31) correspondingly. Moreover, there might exist a static solution $u(\vec{x})$, that describes

a two-soliton configuration and so cannot be consistent with the one-particle representation of the Poincaré group. In this case the relations (29)–(31) obviously do not hold.

On account of these general considerations we show now that the typical hedgehog configurations of nonlinear σ -models describe spinning particles independently of the profile of their chiral angles. In two spatial dimensions, we consider the $O(3)$ σ -model, described by the Lagrangian density (19), with one-particle solution (20). This theory is hoped to reveal the fractional spin and statistics after adding the Hopf term. In the case of 2 spatial dimensions we have two translational $\psi_i^a = \partial_i \varphi^a$, ($i = 1, 2$) and only one rotational $f^a = \varepsilon_{ij} \xi_i \partial_j \varphi^a$ zero modes for each isospin component φ^a . By straightforward substitution it is easy to verify, that the configuration (20) leads to fulfilment of conditions (29)–(31). Thus, the baby-skyrmion solution (20) corresponds to the spinning particle for any choice of the chiral angle.

In 3+1-dimensional space-time, we consider the $SU(2)$ -Skyrme model described by the Lagrangian

$$\mathcal{L} = -\frac{1}{4} \text{tr} L_\mu^2 + \frac{1}{32} \text{tr} [L_\mu L_\nu]^2. \quad (32)$$

In terms of three independent fields ϕ_a the expression (32) can be rewritten as [10]

$$\mathcal{L} = \frac{1}{2} \dot{\phi}_a M_{ab}(\vec{\phi}) \dot{\phi}_b - V(\vec{\phi}) \quad (33)$$

(the definition of $M_{ab}(\vec{\phi})$ and $V(\vec{\phi})$ see in [10, 11]). From the Lagrangian (33) we find the Hamiltonian density

$$\mathcal{H} = \frac{1}{2} \pi_a M_{ab}^{-1}(\vec{\phi}) \pi_b + V(\vec{\phi}), \quad (34)$$

where the canonical field momentum is $\pi_a = M_{ab}(\vec{\phi}) \dot{\phi}_b$.

The treatment of the Skyrme model differs from the theories considered below in that point, that it contains terms of the 4th order in derivatives. As a result, all the scalar products of the model, in particular, the orthogonality conditions (10)–(12) in the vicinity of the static classical solution $\phi_c^a(\vec{\xi})$ acquire a nontrivial integration measure. It is easy to verify, that for the Lagrangian (33) the weight function in the integration measure is $M_{ab}(\phi_c(\vec{\xi}))$.

Now we verify the conditions of particle-likeness for the standard hedgehog configuration

$$\phi_a = \frac{r_a}{r} \phi(r). \quad (35)$$

After some algebra we obtain, that the conditions of particle-likeness for this configuration are satisfied in 3 spatial dimensions as well, and once more it holds independently of the profile of the function $\phi(r)$. Thus, the soliton (35) of the $SU(2)$ -Skyrme model might be embedded into the irreducible representation of the Poincaré group for the particle with spin without any restrictions on the shape of the chiral angle.

To conclude let us mention, that the present analysis can be easily extended to other soliton models including vector fields, etc. On the other hand, the relations (10)–(12) and (29)–(31), being independent of equations of motion, can play an essential role of additional constraints in approximate calculations as well. For example, they can be explored as a test for various sample functions, used in describing the shape of the skyrmion. Concerning the Skyrme model, our analysis is consistent with the well-known result [12], that the spin of $SU(2)$ -skyrmion can be arbitrary.

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